

# ORTHONORMAL WAVELET SYSTEM IN $\ell^2(\mathbb{Z}_N^2)$

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**ABSTRACT.** Using the group theoretic approach based on the set of digits, we first investigate a finite collection of functions in  $\ell^2(\mathbb{Z}_N^2)$  that satisfies some localization properties in a region of the time-frequency plane. The digits are associated with an invertible (expansive/non-expansive) matrix having integer entries. Next, we study and characterize an orthonormal wavelet system for  $\ell^2(\mathbb{Z}_N^2)$ . In addition, some results connecting the uncertainty principle with functions that generate the orthonormal wavelet system having time-frequency localization properties are obtained.

## 1. INTRODUCTION

A time-frequency localized basis plays an important role in extracting both time as well as frequency information of a given signal, and the uncertainty principle helps us to understand how much local information in time and frequency we can extract by using the above mentioned basis. Such basis has many applications in the real life problems, for example, to look closely at a potential tumor in medical image processing, in radar or sonar imaging, for compressing video images in video image analysis, for the recovery of lost part of the signals, etc. Due to this, many researchers are attracted towards the problem of finding good bases having both time as well as frequency localization properties (e.g. [3, 10–12]). Furthermore, in this day and age of computers, processing can be done only when the signal can be stored in memory. Therefore, the importance of discrete and finite signals can not be ignored.

Wavelets are the latest and most successful tools to extract information from many different kinds of data including but certainly not limited to audio signals and images. Our main goal is to study orthonormal wavelet systems having time-frequency localization properties for multidimensional setup. In particular, we want to find conditions on the sets  $I_1, I_2$  and  $A \in GL(2, \mathbb{R})$ , and to characterize  $\Phi = \{\varphi_p\}_{p \in I_2} \subset \ell^2(\mathbb{Z}_N^2)$  such that the collection  $\mathfrak{B}(\Phi)$  (with distinct elements) defined by

$$\mathfrak{B}(\Phi) := \{T_{Ak}\varphi_p : k \in I_1 \subseteq \mathbb{Z}_N^2, p \in I_2 \subset \mathbb{N}\} \quad (1.1)$$

forms an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  having time-frequency localization properties, where for each  $k \in \mathbb{Z}_N^2$  and  $f \in \ell^2(\mathbb{Z}_N^2)$ , the *translation operator*  $T_k$  on  $\ell^2(\mathbb{Z}_N^2)$  is defined by  $T_k(f)(m) = f(m - k)$ , for all  $m \in \mathbb{Z}_N^2$ . For  $N \in \mathbb{N}$ , the space  $\ell^2(\mathbb{Z}_N^2)$  denotes the collection of all  $N \times N$  complex matrices by identifying  $f \in \ell^2(\mathbb{Z}_N^2)$  with the  $N \times N$  matrix  $(f(n_1, n_2)^t)_{n_1, n_2 \in \mathbb{Z}_N}$ . Note that it is an  $N^2$ -dimensional Hilbert space with usual inner product

$$\langle f, g \rangle = \text{trace}(g^* f) = \sum_{(n_1, n_2)^t \in \mathbb{Z}_N^2} f(n_1, n_2)^t \overline{g(n_1, n_2)^t}, \quad \text{for all } f, g \in \ell^2(\mathbb{Z}_N^2),$$

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where  $g^*$  and  $\mathbb{Z}_N := \{0, 1, 2, \dots, N-1\}$  denote the conjugate transpose of  $g$  and a group of integers under addition modulo  $N$ , respectively, and  $\mathbb{Z}_N^2 = \mathbb{Z}_N \times \mathbb{Z}_N = \{(m, n)^t := \begin{pmatrix} m \\ n \end{pmatrix} : m, n \in \mathbb{Z}_N\}$ .

By a *time-frequency localized basis*, we mean that every vector in the basis is time localized as well as frequency localized. An  $f \in \ell^2(\mathbb{Z}_N^2)$  is *time localized* near  $n_0 \in \mathbb{Z}_N^2$  if most of the components  $f(n)$  of  $f$  are 0 or relatively small, except for a few values of  $n$  close to  $n_0$  and it is said to be *frequency localized* near  $n_0$  if most of the components of discrete Fourier transform of  $f$  are 0 or relatively small, except for a few values of  $n$  close to  $n_0$ . At this juncture, it is pertinent to note that the standard orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  is time localized but not frequency localized, while its Fourier basis is frequency localized but not time localized. In the last two decades a lot of research on time-frequency analysis has been done by several authors for the various spaces, namely, finite and infinite abelian groups, Euclidean spaces, etc. (e.g. [3–8, 11–13]).

In order to compare  $\mathfrak{B}(\Phi)$  defined by (1.1) with the classical notion of orthonormal multiwavelet [13], let us consider  $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subset L^2(\mathbb{R}^n)$  with the dilation matrix  $A_0$  and translation  $\mathbb{Z}^n$ , which provides an orthonormal basis  $\mathcal{A}(\Psi)$  for  $L^2(\mathbb{R}^n)$ , where

$$\mathcal{A}(\Psi) := \{\psi_{j,k}^l := |\det(A_0)|^{j/2} \psi_l(A_0^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n \text{ and } \psi_l \in \Psi\}.$$

In view of the system  $\mathcal{A}(\Psi)$ , note that elements of  $\mathfrak{B}(\Phi)$  can be written as  $\varphi_{0,Ak}^p = T_{Ak} \varphi_p$ , where  $p \in I_2, k \in I_1$ . Further, we remark that if  $\mathfrak{B}(\Phi)$  is an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ , then one cannot construct an orthogonal set by adding elements into  $\mathfrak{B}(\Phi)$ . This, in turn, implies that there is no need to use dilation operator in the definition of an orthonormal wavelet system (ONWS) in  $\ell^2(\mathbb{Z}_N^2)$ . Thus, it leads us to conclude that we cannot use the natural definition of orthonormal multiwavelet of  $L^2(\mathbb{R}^n)$  in our case.

Further, in this paper, we study Heisenberg uncertainty principle in terms of the pair of representations of a given signal by coupling ONWS with the standard basis and Fourier basis. This study can be useful to provide uniqueness properties of the sparse representation of the signal. The theory of uncertainty principle related to the pair of bases and its applications has been developed by many authors (e.g. [1, 2, 4, 8, 9, 11]).

The remainder of the article is organized as follows: In Section 2, we make a necessary background to investigate a finite collection of functions in  $\ell^2(\mathbb{Z}_N^2)$  that satisfies some localization properties in a region of the time-frequency plane using the group theoretic approach based on the set of digits. Using these functions, we first introduce an orthonormal wavelet system in  $\ell^2(\mathbb{Z}_N^2)$  and then provide a characterization for its generators. Section 3 presents some results on uncertainty principle corresponding to this orthonormal wavelet system. The results obtained in Sections 2 and 3 for  $\mathbb{Z}_N^2$  can be generalized for  $\mathbb{Z}_N^M$ ,  $M \in \mathbb{N}$ .

Throughout the paper,  $\mathcal{M}(2, X)$  and  $GL(2, X)$  denote the collection of all  $2 \times 2$  matrices and collection of all  $2 \times 2$  invertible matrices over  $X$ , respectively, where  $X$  is used as  $\mathbb{Z}_N$  or  $\mathbb{Z}$  (the set of integers) or  $\mathbb{R}$  (the set of real numbers). The notation  $|\cdot|$  denotes the cardinality of a set, or, the absolute value of a complex number. For  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}(2, \mathbb{Z})$  and  $I \subseteq \mathbb{Z}_N^2$ ,

$$AI := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \bmod N = \begin{pmatrix} an_1 \oplus bn_2 \\ cn_1 \oplus dn_2 \end{pmatrix} : (n_1, n_2)^t \in I \right\} \subset \mathbb{Z}_N^2,$$

where  $\oplus$  denotes addition modulo  $N$ .

2. GROUP THEORETIC APPROACH FOR ONWS IN  $\ell^2(\mathbb{Z}_N^2)$ 

Our main motive in this section is to answer the question imposed for the system  $\mathfrak{B}(\Phi)$  defined in (1.1). For this, we need to recall and establish some group theoretic results. From (1.1), it is clear that  $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$ , and  $|\mathfrak{B}(\Phi)| = |AI_1||I_2| \leq |I_1||I_2|$ . Further, we note that the system  $\mathfrak{B}(\Phi)$  should have  $N^2$  distinct elements to become an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ . Hence, we have  $N^2 = |\mathfrak{B}(\Phi)| = |AI_1||I_2|$ , which implies that  $|AI_1| = N^2/|I_2|$ . To find  $I_1$  and  $I_2$ , firstly we are interested in computing  $|AI_1|$ , where  $I_1 \subseteq \mathbb{Z}_N^2$ . Clearly,  $A\mathbb{Z}_N^2 \subseteq \mathbb{Z}_N^2$ . Now, by using the fact that  $\mathbb{Z}_N^2$  is an abelian group, we conclude that  $A\mathbb{Z}_N^2$  is a normal subgroup of  $\mathbb{Z}_N^2$ . The motivation for considering subgroups of  $\mathbb{Z}_N^2$  of this type comes from the following theorem:

**Theorem 2.1.** *Every subgroup of  $\mathbb{Z}^2$  is of the form  $A\mathbb{Z}^2$  for some  $A \in \mathcal{M}(2, \mathbb{Z})$ .*

*Proof.* We start by claiming that every subgroup of  $\mathbb{Z}^2$  is generated by at most two generators. This means, for some integer  $n \geq 1$ , we have to show that a subgroup of  $\mathbb{Z}^2$ , which is generated by  $n$  generators is actually generated by at most two. For this, it is enough to assume three elements  $(a, b)^t, (c, d)^t, (e, f)^t \in \mathbb{Z}^2$ , which are supposed to generate a subgroup  $H$  of  $\mathbb{Z}^2$  and we need to show that one of them is redundant. For this, let  $a, c \neq 0$ . Then, we choose  $x, y \in \mathbb{Z}$  such that  $x(a, b)^t + y(c, d)^t = (0, b')^t$  for some  $b' \in \mathbb{Z}$ . Clearly,  $(0, b')^t, (c, d)^t$  and  $(e, f)^t$  will generate  $H$ . If  $b' = 0$ , then the result follows. Suppose  $b' \neq 0$ . By proceeding in the similar way, choose  $x', y' \in \mathbb{Z}$  such that  $x'(c, d)^t + y'(e, f)^t = (e', 0)^t$  for some  $e' \in \mathbb{Z}$ . Then,  $(0, b')^t, (c, d)^t$  and  $(e', 0)^t$  will generate  $H$ , and hence, we can select  $m, n, p \in \mathbb{Z}$  satisfying the equation  $m(0, b')^t + n(c, d)^t + p(e', 0)^t = (0, 0)^t$ , for which one choice is  $m = -e'd, n = e'b'$ , and  $p = -b'c$ . Therefore,  $m(0, b')^t + n(c, d)^t + p(e', 0)^t, (c, d)^t$  and  $(e', 0)^t$  are also generators of  $H$ , which implies that  $H$  is generated by  $(c, d)^t$  and  $(e', 0)^t$ . Combining the facts discussed above, we get  $H = \langle (a, b)^t, (c, d)^t, (e, f)^t \rangle = \langle (c, d)^t, (e', 0)^t \rangle$ . Further note that, if  $(c, d)^t$  and  $(e', 0)^t$  are linearly independent, then  $H$  is generated by two generators, and hence we can write

$$H = \langle (c, d)^t, (e', 0)^t \rangle = \left\{ \begin{pmatrix} c & e' \\ d & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} : m, n \in \mathbb{Z} \right\} = \begin{pmatrix} c & e' \\ d & 0 \end{pmatrix} \mathbb{Z}^2 = \begin{pmatrix} e' & c \\ 0 & d \end{pmatrix} \mathbb{Z}^2.$$

Otherwise,  $H$  will be generated by only one generator say  $(c, d)^t$ , and then it can be written as

$$H = \langle (c, d)^t \rangle = \{(cm, dm)^t : m \in \mathbb{Z}\} = \left\{ \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} : m, n \in \mathbb{Z} \right\} = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \mathbb{Z}^2 = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \mathbb{Z}^2.$$

Hence, we conclude the result from the fact that in each case, there exists  $A \in \mathcal{M}(2, \mathbb{Z})$  such that  $H = A\mathbb{Z}^2$ .  $\square$

For a  $2 \times 2$  expansive matrix  $A$  having integer entries, it is well known that the order of group  $\frac{\mathbb{Z}^2}{A\mathbb{Z}^2}$  is  $|\det(A)|$  [13, Proposition 5.5] while the result is also true for any invertible matrix with integer entries. By an *expansive matrix*, we mean that all of its eigenvalues  $\lambda$  satisfy  $|\lambda| > 1$ . Now, we are interested in finding the order of group  $A\mathbb{Z}_N^2$ . Therefore, we have the following result:

**Theorem 2.2.** *Let  $N \in \mathbb{N}$  and  $A \in GL(2, \mathbb{R})$  such that  $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$ , and let  $B \in GL(2, \mathbb{Z})$ , where  $B = NA^{-1}$ . Then, the determinant  $\det(A)$  of  $A$  divides  $N^2$ , and hence the order of groups  $A\mathbb{Z}_N^2$  and  $\frac{\mathbb{Z}_N^2}{A\mathbb{Z}_N^2}$  is given by  $|A\mathbb{Z}_N^2| = |\det(B)| = \frac{N^2}{|\det(A)|}$  and  $\left| \frac{\mathbb{Z}_N^2}{A\mathbb{Z}_N^2} \right| = |\det(A)|$ , respectively.*

*Proof.* Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$  be such that  $a, b, c, d \in \mathbb{Z}$ . Then,  $B = NA^{-1} \in GL(2, \mathbb{Z})$  is given by  $\begin{pmatrix} \frac{Nd}{\det(A)} & \frac{-Nb}{\det(A)} \\ \frac{-Nc}{\det(A)} & \frac{Na}{\det(A)} \end{pmatrix}$ . Since  $\frac{N^2}{\det(A)} = \frac{Na}{\det(A)} \times \frac{Nd}{\det(A)} - \frac{Nb}{\det(A)} \times \frac{Nc}{\det(A)}$  is an integer, the  $\det(A)$  divides  $N^2$ . Next, we have to show that  $|AZ_N^2| = |\det(B)| = \frac{N^2}{|\det(A)|}$ . For this, it is enough to see isomorphism between groups  $A\mathbb{Z}_N^2$  and  $\frac{\mathbb{Z}^2}{B\mathbb{Z}^2}$ , that means,  $A\mathbb{Z}_N^2 \cong \frac{\mathbb{Z}^2}{B\mathbb{Z}^2}$ , which implies  $|AZ_N^2| = \left| \frac{\mathbb{Z}^2}{B\mathbb{Z}^2} \right| = |\det(B)| = \frac{N^2}{|\det(A)|}$ , and hence  $\left| \frac{\mathbb{Z}_N^2}{AZ_N^2} \right| = \frac{|\mathbb{Z}_N^2|}{|AZ_N^2|} = |\det(A)|$ . Now, in order to prove the above claim, consider a map  $f : \mathbb{Z}^2 \rightarrow A\mathbb{Z}_N^2$  defined for all  $(n_1, n_2)^t \in \mathbb{Z}^2$  by  $f(n_1, n_2)^t = A \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \bmod N$ , where  $n_i \equiv r_i \bmod N$ , for  $i = 1, 2$ . Clearly,  $f$  is well defined and further, the homomorphism of  $f$  follows by noting that for all  $(n_1, n_2)^t, (m_1, m_2)^t \in \mathbb{Z}^2$ , we can write  $n_i \equiv r_i \bmod N, m_j \equiv R_j \bmod N$ , and  $(n_i + m_j) \equiv R_{ij} \bmod N$ , for  $i, j = 1, 2$ , and hence

$$\begin{aligned} f((n_1, n_2)^t + (m_1, m_2)^t) &= A \begin{pmatrix} R_{11} \\ R_{22} \end{pmatrix} \bmod N = A \begin{pmatrix} r_1 \oplus R_1 \\ r_2 \oplus R_2 \end{pmatrix} \bmod N \\ &= A \left( \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \oplus \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \right) \bmod N = f(n_1, n_2)^t \oplus f(m_1, m_2)^t. \end{aligned}$$

Next, the map  $f$  is onto and its kernel  $\text{Ker}(f)$  is given by

$$\begin{aligned} \text{Ker}(f) &= \left\{ (m_1, m_2)^t \in \mathbb{Z}^2 : f(m_1, m_2)^t = A \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \bmod N = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ (m_1, m_2)^t \in \mathbb{Z}^2 : \text{for } q_1, q_2 \in \mathbb{Z}, A \begin{pmatrix} m_1 - Nq_1 \\ m_2 - Nq_2 \end{pmatrix} \bmod N = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ (m_1, m_2)^t \in \mathbb{Z}^2 : A \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in N\mathbb{Z} \times N\mathbb{Z} = NI\mathbb{Z}^2 \right\} \\ &= \left\{ (m_1, m_2)^t \in \mathbb{Z}^2 : (m_1, m_2)^t \in NA^{-1}\mathbb{Z}^2 = B\mathbb{Z}^2 \right\} \\ &= B\mathbb{Z}^2, \text{ where } I \text{ is a } 2 \times 2 \text{ identity matrix.} \end{aligned}$$

Therefore, by fundamental theorem of group homomorphism,  $A\mathbb{Z}_N^2 \cong \frac{\mathbb{Z}^2}{B\mathbb{Z}^2}$ . □

**Remark 2.3.** If  $A \in GL(2, \mathbb{R})$  is a matrix such that  $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$  and  $\det(A)$  divides  $N^2$ , then the matrix  $NA^{-1}$  need not be a member of  $GL(2, \mathbb{Z})$ . For example, let  $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$  and  $N = 4$ . Then  $\det(A)$  divides  $N^2$ , but  $NA^{-1} = \frac{4}{8} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix} \notin GL(2, \mathbb{Z})$ .

From Theorem 2.2, we have  $\left| \frac{\mathbb{Z}_N^2}{B\mathbb{Z}_N^2} \right| = |\det(B)| = \frac{N^2}{|\det(A)|} = r$ , (say). In the rest of the paper, by  $\{\bar{\beta}_i\}_{i=1}^r$  we denote the distinct coset representatives of  $\frac{\mathbb{Z}_N^2}{B\mathbb{Z}_N^2}$ , where for  $1 \leq i \leq r$ ,  $\bar{\beta}_i = \beta_i + B\mathbb{Z}_N^2$  such

that  $\beta_i \in \mathfrak{D}$ . Here, the set  $\mathfrak{D} \subseteq (\mathbb{Z}_N^2 \setminus B\mathbb{Z}_N^2) \cup \{(0,0)^t\}$  is termed as *the set of digits* corresponding to the cosets of  $B\mathbb{Z}_N^2$  in  $\mathbb{Z}_N^2$ . Clearly, for  $1 \leq i \neq j \leq r$ , we have  $\beta_i \neq \beta_j$  and  $(\beta_i + B\mathbb{Z}_N^2) \cap (\beta_j + B\mathbb{Z}_N^2) = \phi$ . Hence,  $|\mathfrak{D}| = \left| \frac{\mathbb{Z}_N^2}{B\mathbb{Z}_N^2} \right|$  and we can write,  $\mathbb{Z}_N^2 = \bigcup_{\beta \in \mathfrak{D}} (\beta + B\mathbb{Z}_N^2)$ . The set  $\mathfrak{D}$  satisfies some nice properties, which are discussed in the following result:

**Theorem 2.4.** *With the assumption of Theorem 2.2, let us consider the set  $\mathfrak{D} \subseteq \mathbb{Z}_N^2$  defined as above with  $(0,0)^t$  element having property that  $Ad_1 \neq Ad_2$  whenever  $d_1 \neq d_2 \in \mathfrak{D}$ . Then, we can choose  $\mathfrak{D}$  satisfying following properties:*

- (i)  $|\mathfrak{D}| = |A\mathbb{Z}_N^2| = |A(\mathfrak{D})| = \frac{N^2}{|\det(A)|}$ . (ii)  $A\mathbb{Z}_N^2 = A(\mathfrak{D})$ .
- (iii) For some  $\beta \in \mathfrak{D}$ ,  $A\beta = (0,0)^t$  implies that  $\beta = (0,0)^t$ .
- (iv) For  $\bar{\beta}_1, \bar{\beta}_2 \in \left\{ \bar{\beta} = \beta + B\mathbb{Z}_N^2 : \beta \in \mathfrak{D} \right\}$ ,  $\bar{\beta}_1 \cap \bar{\beta}_2 = \phi$ , and we can write  $\mathbb{Z}_N^2 = \bigcup_{\beta \in \mathfrak{D}} (\beta + B\mathbb{Z}_N^2)$ .
- (v) For  $\beta_1, \beta_2 \in \mathfrak{D}$ , we have  $(\beta_1 + \beta_2) + B\mathbb{Z}_N^2 = \beta + B\mathbb{Z}_N^2$  for some  $\beta \in \mathfrak{D}$ .
- (vi) For  $\beta_1, \beta_2 \in \mathfrak{D}$ , there exists  $\beta \in \mathfrak{D}$  such that  $A\beta_1 - A\beta_2 = A\beta$ . Moreover,  $\beta_1 = \beta_2$  if and only if  $\beta = (0,0)^t$ .

*Proof.* The explanation for the above properties is as follows:

- (i) From Theorem 2.2 and above discussion about the set of digits, we have  $|\mathfrak{D}| = \left| \frac{\mathbb{Z}_N^2}{B\mathbb{Z}_N^2} \right| = |\det(B)| = \frac{N^2}{|\det(A)|} = |A\mathbb{Z}_N^2|$ , and hence,  $|\mathfrak{D}| = |A(\mathfrak{D})|$  by noting the fact that  $Ad_1 \neq Ad_2$  whenever  $d_1 \neq d_2 \in \mathfrak{D}$ .
- (ii) The result  $A\mathbb{Z}_N^2 = A(\mathfrak{D})$  follows by noting that  $A(\mathfrak{D}) \subseteq A\mathbb{Z}_N^2$  and  $|A(\mathfrak{D})| = |A\mathbb{Z}_N^2|$ .
- (iii) For this part, let us assume by contrary that there exists some  $\beta_1 \neq (0,0)^t \in \mathfrak{D}$  such that  $A\beta_1 = (0,0)^t$ . But it is given that  $\beta_1 \neq (0,0)^t \in \mathfrak{D}$  implies  $A\beta_1 \neq (0,0)^t$ , which is a contradiction.
- (iv) The result is obvious which follows from the definition of  $\mathfrak{D}$ .
- (v) For  $\beta_1, \beta_2 \in \mathfrak{D}$ , we have  $(\beta_1 + B\mathbb{Z}_N^2) + (\beta_2 + B\mathbb{Z}_N^2) = (\beta_1 + \beta_2) + B\mathbb{Z}_N^2 \in \frac{\mathbb{Z}_N^2}{B\mathbb{Z}_N^2}$  and hence, we can write  $(\beta_1 + \beta_2) + B\mathbb{Z}_N^2 = \beta + B\mathbb{Z}_N^2$  for some  $\beta \in \mathfrak{D}$ .
- (vi) In view of the fact  $A(\mathfrak{D}) = A\mathbb{Z}_N^2$ , and for  $\beta_1, \beta_2 \in \mathfrak{D}$ , we have  $A(\beta_1) - A(\beta_2) \in A\mathbb{Z}_N^2$ , and hence there exists  $\beta \in \mathfrak{D}$  such that  $A(\beta_1) - A(\beta_2) = A\beta$ . For the remaining part, let  $\beta_1 = \beta_2$ . Then  $A(\beta_1) = A(\beta_2)$ , which implies that  $A(\beta_1) - A(\beta_2) = A(\beta) = (0,0)^t$ , and hence  $\beta = (0,0)^t$ . Conversely, suppose  $\beta = (0,0)^t$ . This means,  $A(\beta) = A(\beta_1) - A(\beta_2) = (0,0)^t$ , which says that either  $\beta_1 = \beta_2$  or  $A(\beta_1) = A(\beta_2)$ , but  $A(\beta_1) = A(\beta_2)$  is possible only for  $\beta_1 = \beta_2$ .

□

In order to understand about  $\mathfrak{D}$ , we provide following example:

**Example 2.5.** Let  $N = 2$  and  $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}$ . Then,  $A\mathbb{Z}_2^2 = B\mathbb{Z}_2^2 = \{(0,0)^t, (0,1)^t\}$ , where  $B = NA^{-1} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ , and hence  $|A\mathbb{Z}_2^2| = |B\mathbb{Z}_2^2| = 2$  and  $\left| \frac{\mathbb{Z}_2^2}{B\mathbb{Z}_2^2} \right| = |\det(B)| = 2$ . Further, we have

$$(0,0)^t + B\mathbb{Z}_2^2 = \overline{(0,0)^t} = \overline{(0,1)^t} = \{(0,0)^t, (0,1)^t\}, \overline{(1,0)^t} = \overline{(1,1)^t} = \{(1,0)^t, (1,1)^t\},$$

and hence, we can choose  $\mathfrak{D} = \{(0,0)^t, (1,0)^t\} \subset \mathbb{Z}_2^2$ , where  $(0,0)^t \in \mathfrak{D}$  and  $Ad_1 \neq Ad_2$  whenever  $d_1 \neq d_2 \in \mathfrak{D}$ . Now, it can be easily checked that  $\mathfrak{D}$  satisfies all properties of Theorem 2.4.

In the rest of paper, we assume that *the set of digits  $\mathfrak{D}$  satisfies properties of Theorem 2.4*. Further, by the set  $\mathfrak{D}^* \subseteq (\mathbb{Z}_N^2 \setminus C\mathbb{Z}_N^2) \cup \{(0,0)^t\}$ , we denote *the set of digits* corresponding to the cosets of  $C\mathbb{Z}_N^2$  in  $\mathbb{Z}_N^2$ , where  $C = B^t =:$  transpose of the matrix  $B$ . Note that  $|\mathfrak{D}^*| = |\mathfrak{D}|$  and we can write  $\mathbb{Z}_N^2 = \bigcup_{m \in \mathfrak{D}^*} (m + C\mathbb{Z}_N^2)$ .

Now, we are in position to provide an answer about the question for the system  $\mathfrak{B}(\Phi)$  defined by (1.1). Theorem 2.4 implies that  $|A(\mathfrak{D})| = |\mathfrak{D}| = \frac{N^2}{|\det(A)|}$  and from (1.1), we have  $|\mathfrak{B}(\Phi)| = |AI_1||I_2|$ . In order to be an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ ,  $\mathfrak{B}(\Phi)$  should have  $N^2$  distinct elements forming orthonormal set, hence  $N^2 = |\mathfrak{B}(\Phi)| = |AI_1||I_2|$ . Further, by assuming  $I_1 = \mathfrak{D}$ , we get  $N^2 = |A(\mathfrak{D})||I_2|$ , which implies  $|I_2| = |\det(A)|$  in view of the fact that  $|AI_1| = |A(\mathfrak{D})| = \frac{N^2}{|\det(A)|}$ . Now, first we consider a collection whose elements are translations of a single vector  $\varphi_0 \in \ell^2(\mathbb{Z}_N^2)$ . Then, we have  $|I_2| = 1$ , that is,  $|\det(A)| = 1$ , which implies  $|AI_1| = N^2$ , and hence in this case, we must have  $I_1 = \mathfrak{D} = \mathbb{Z}_N^2$ . Therefore, the system defined in (1.1) takes the following form:

$$\mathfrak{B}(\varphi_0) := \{T_{Ak}\varphi_0 : k \in \mathbb{Z}_N^2\} = \{T_k\varphi_0 : k \in \mathbb{Z}_N^2\} \subset \ell^2(\mathbb{Z}_N^2). \quad (2.1)$$

Next, we want to characterize  $\varphi_0$  such that the system defined in (2.1) yields an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  having time-frequency localization properties. For this, we need following definition and result:

The *discrete Fourier transform* (DFT) and *Inverse discrete Fourier transform* (IDFT) on  $\ell^2(\mathbb{Z}_N^2)$  are defined for  $m, n \in \mathbb{Z}_N^2$ , and  $f, g \in \ell^2(\mathbb{Z}_N^2)$  by

$$\widehat{f}(m) = \sum_{s \in \mathbb{Z}_N^2} f(s) e^{-2\pi i \langle m, s \rangle / N} \text{ and } g^\vee(n) = \frac{1}{N^2} \sum_{l \in \mathbb{Z}_N^2} g(l) e^{2\pi i \langle n, l \rangle / N},$$

respectively. Further, we have  $(\widehat{f})^\vee(n) = f(n)$ , for all  $n \in \mathbb{Z}_N^2$  and the Plancherel's formula

$$\langle f, g \rangle = \frac{1}{N^2} \sum_{m \in \mathbb{Z}_N^2} \widehat{f}(m) \overline{\widehat{g}(m)} = \frac{1}{N^2} \langle \widehat{f}, \widehat{g} \rangle,$$

which provides Parseval's formula for  $f = g$ .

The following lemma can be proved easily:

**Lemma 2.6.** *For  $A \in \mathcal{M}(2, \mathbb{Z})$ ,  $k, k_1 \in \mathbb{Z}_N^2$  and  $f, g \in \ell^2(\mathbb{Z}_N^2)$ , we have following properties:*

- (i)  $(\widehat{T_{Ak}f})(m) = e^{-2\pi i \langle m, Ak \rangle / N} \widehat{f}(m)$  for all  $m \in \mathbb{Z}_N^2$ .
- (ii)  $\langle T_{Ak_1}f, T_{Ak}g \rangle = \langle f, T_{(Ak-Ak_1)}g \rangle$ .
- (iii)  $\widehat{\delta}(k) = 1$ , where  $\delta(k) = 1$  for  $k = (0,0)^t$ , and zero otherwise.

In order to be an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ , the system  $\mathfrak{B}(\varphi_0)$  as defined in (2.1) has to satisfy following condition:



**Theorem 2.7.** *The system  $\mathfrak{B}(\varphi_0)$  forms an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  if and only if we have  $|\widehat{\varphi_0}(k)|^2 = 1$  for all  $k \in \mathbb{Z}_N^2$ . Moreover, orthonormal basis of this form is not frequency localized.*

*Proof.* The system  $\mathfrak{B}(\varphi_0)$  forms an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  if and only if  $\delta(k) = \langle \varphi_0, T_k \varphi_0 \rangle$ , for  $k \in \mathbb{Z}_N^2$ . By using the Plancherel's formula, we can write

$$\delta(k) = \frac{1}{N^2} \langle \widehat{\varphi_0}, \widehat{T_k \varphi_0} \rangle = \frac{1}{N^2} \sum_{n \in \mathbb{Z}_N^2} |\widehat{\varphi_0}(n)|^2 e^{2\pi i \langle n, k \rangle / N} = \mathcal{G}^\vee(k),$$

where  $\mathcal{G}(n) = |\widehat{\varphi_0}(n)|^2$ . Hence, the system  $\mathfrak{B}(\varphi_0)$  forms an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  if and only if  $\delta(k) = \mathcal{G}^\vee(k)$ , which is if and only if  $|\widehat{\varphi_0}(k)|^2 = 1$  for all  $k \in \mathbb{Z}_N^2$ , in view of Lemma 2.6. This implies that the vector  $\varphi_0$  is not frequency localized, and hence the orthonormal basis of the form  $\mathfrak{B}(\varphi_0)$  is not frequency localized.  $\square$

In order to get time as well as frequency localized orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ , we modify our approach by considering a collection whose elements are translations of two or more vectors. This means, we consider the case when  $|I_2| = |\det(A)| > 1$ . From Theorem 2.4, we have  $|AI_1| = |A(\mathfrak{D})| = \frac{N^2}{|\det(A)|}$ , and hence in this case, we must have  $I_1 = \mathfrak{D} \subsetneq \mathbb{Z}_N^2$ . Therefore, for  $|\det(A)| = q \geq 2$  and  $\Phi = \{\varphi_p\}_{p=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^2)$ , the system in (1.1) can be redefined as:

$$\mathfrak{B}(\Phi) := \{T_{Ak} \varphi_p : k \in \mathfrak{D}, 0 \leq p \leq q-1\} \subset \ell^2(\mathbb{Z}_N^2). \quad (2.2)$$

**Remark 2.8.** An important point that one should keep in mind while choosing  $\Phi$  in (2.2) is to select  $\Phi$  in such a way that there should not exist any element  $\psi \in \ell^2(\mathbb{Z}_N^2)$  such that  $\{T_k \psi : k \in \mathbb{Z}_N^2\} = \mathfrak{B}(\Phi)$ , otherwise  $\mathfrak{B}(\Phi)$  will become similar with the system (2.1), and Theorem 2.7 says that orthonormal basis of this form is not frequency localized. For a better explanation of this fact, we have the following result which is also useful in the sequel:

**Proposition 2.9.** *Let  $\{e_m\}_{m \in \mathbb{Z}_N^2}$  be a standard orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ , where for each  $j \in \mathbb{Z}_N^2$ , we define  $e_j(n) = 1$  if  $n = j$  and 0 otherwise. Then, for  $N \in \mathbb{N}$  and  $A \in GL(2, \mathbb{R})$  such that  $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$  with  $|\det(A)| = q \geq 2$ , there exists  $\{\gamma_j\}_{j=0}^{q-1} \subset \mathbb{Z}_N^2$  for which  $\{T_{Ak} e_{\gamma_j} : k \in \mathfrak{D}, 0 \leq j \leq q-1\} = \{T_k e_n\}_{k \in \mathbb{Z}_N^2}$ , for some  $n \in \mathbb{Z}_N^2$ .*

*Proof.* Let  $n \in \mathbb{Z}_N^2$ . Then, both the collections  $\{T_k e_n\}_{k \in \mathbb{Z}_N^2}$  and  $\{e_m\}_{m \in \mathbb{Z}_N^2}$  are same, and hence we claim that  $\{T_{Ak} e_{\gamma_j} : k \in \mathfrak{D}, 0 \leq j \leq q-1\} = \{e_m\}_{m \in \mathbb{Z}_N^2}$ . For this, it is enough to show that there exists  $\{\gamma_j\}_{j=0}^{q-1} \subset \mathbb{Z}_N^2$  such that the collection  $\{\gamma_j + Ak : k \in \mathfrak{D}, 0 \leq j \leq q-1\} = \mathbb{Z}_N^2$ , since for  $k \in \mathfrak{D}$  and  $0 \leq j \leq q-1$ , we have  $T_{Ak} e_{\gamma_j} = e_{(\gamma_j + Ak)}$ . Now, from Theorem 2.2, we have

$\left| \frac{\mathbb{Z}_N^2}{A\mathbb{Z}_N^2} \right| = |\det(A)| = q$ , and hence, we have  $q$  distinct coset representatives of  $A\mathbb{Z}_N^2$  in  $\mathbb{Z}_N^2$ , say,  $\{\bar{\alpha}_j\}_{j=0}^{q-1}$ ,

where for  $0 \leq j \leq q-1$ , we define  $\bar{\alpha}_j = \alpha_j + A\mathbb{Z}_N^2$ . Here, the set  $\{\alpha_j\}_{j=0}^{q-1} =: \mathfrak{D}_0 \subseteq (\mathbb{Z}_N^2 \setminus A\mathbb{Z}_N^2) \cup \{(0, 0)^t\}$  has following properties:  $\alpha_{j_1} \neq \alpha_{j_2}$  and  $(\alpha_{j_1} + A\mathbb{Z}_N^2) \cap (\alpha_{j_2} + A\mathbb{Z}_N^2) = \emptyset$ , for  $0 \leq j_1 \neq j_2 \leq q-1$ . Hence,

$|\mathfrak{D}_0| = \left| \frac{\mathbb{Z}_N^2}{A\mathbb{Z}_N^2} \right|$  and we can write,  $\mathbb{Z}_N^2 = \bigcup_{\alpha \in \mathfrak{D}_0} (\alpha + A\mathbb{Z}_N^2) = \{\alpha + Ak : \alpha \in \mathfrak{D}_0, k \in \mathfrak{D}\}$ . This implies

that there exists  $\{\alpha_j\}_{j=0}^{q-1} = \mathfrak{D}_0 \subset \mathbb{Z}_N^2$  such that  $\{T_{Ak} e_{\alpha_j} : k \in \mathfrak{D}, 0 \leq j \leq q-1\} = \{e_m\}_{m \in \mathbb{Z}_N^2}$ .  $\square$

Next, we are defining an orthonormal wavelet system for  $\ell^2(\mathbb{Z}_N^2)$ :

**Definition 2.10. Orthonormal Wavelet System:** We call the system  $\mathfrak{B}(\Phi)$ , defined in (2.2), an *orthonormal wavelet system (ONWS)* for  $\ell^2(\mathbb{Z}_N^2)$ , if it forms an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$ .

Following is our main result of this section which characterizes  $\Phi \subset \ell^2(\mathbb{Z}_N^2)$  such that the system  $\mathfrak{B}(\Phi)$  defined in (2.2) provides an ONWS for  $\ell^2(\mathbb{Z}_N^2)$ :

**Theorem 2.11.** *Let  $A \in GL(2, \mathbb{R})$  be a matrix such that  $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$  and let  $C = (NA^{-1})^t \in GL(2, \mathbb{Z})$  for some  $N \in \mathbb{N}$ . Consider  $\Phi = \{\varphi_p\}_{p=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^2)$ , where  $|\det(A)| = q \geq 2$ . Then, the following statements are equivalent:*

- (i)  $\mathfrak{B}(\Phi) \subset \ell^2(\mathbb{Z}_N^2)$  forms an ONWS in  $\ell^2(\mathbb{Z}_N^2)$ .
- (ii) For all  $k \in \mathfrak{D}$  and  $(p_1, p_2) \in \{(m, n) : 0 \leq m \leq n \leq q-1\}$ ,

$$\sum_{\gamma \in C\mathbb{Z}_N^2} \widehat{\varphi}_{p_1}(k + \gamma) \overline{\widehat{\varphi}_{p_2}(k + \gamma)} = q\delta_{p_1 p_2}.$$

- (iii) The system matrix  $\mathcal{S}_\Phi(k)$  of  $\Phi$  is unitary for each  $k \in \mathfrak{D}$ , where

$$\mathcal{S}_\Phi(k) = \frac{1}{\sqrt{q}} \left( \widehat{\varphi}_p(k + \gamma) \right)_{\substack{\gamma \in C\mathbb{Z}_N^2 \\ 0 \leq p \leq q-1}}.$$

For this, we need to set up the background in the form of following results:

**Lemma 2.12.** *For  $\alpha \in A\mathbb{Z}_N^2$  and  $\beta \in C\mathbb{Z}_N^2$ , the inner product  $\langle \alpha, \beta \rangle \in N\mathbb{Z}$ , where  $A$  and  $C$  are as defined in Theorem 2.11.*

*Proof.* Let  $\alpha = (\alpha_1, \alpha_2)^t \in A\mathbb{Z}_N^2$  and  $\beta = (\beta_1, \beta_2)^t \in C\mathbb{Z}_N^2$ . Then, there exist  $n = (n_1, n_2)^t$  and  $m = (m_1, m_2)^t \in \mathbb{Z}_N^2$  such that  $\alpha = An$  and  $\beta = Cm$ , where  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$  is a matrix

with  $A\mathbb{Z}^2 \subseteq \mathbb{Z}^2$  and  $C = \begin{pmatrix} \frac{Nd}{\det(A)} & \frac{-Nc}{\det(A)} \\ \frac{-Nb}{\det(A)} & \frac{Na}{\det(A)} \end{pmatrix} \in GL(2, \mathbb{Z})$ . Now, the result follows by observing that

$$\langle \alpha, \beta \rangle = \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \begin{pmatrix} \frac{Nd}{\det(A)} & \frac{-Nc}{\det(A)} \\ \frac{-Nb}{\det(A)} & \frac{Na}{\det(A)} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \right\rangle = Np, \text{ for } p = n_1 m_1 + n_2 m_2 \in \mathbb{Z}. \quad \square$$

**Proposition 2.13.** *Let  $\{E_k\}_{k \in \mathfrak{D}} \subseteq \ell^2(\mathfrak{D}^*)$ , where for each  $k \in \mathfrak{D}$  and  $A$  as defined in Theorem 2.11, we define  $E_k(m) = \frac{1}{\sqrt{|\mathfrak{D}^*|}} e^{2\pi i \langle m, Ak \rangle / N}$  for all  $m \in \mathfrak{D}^*$ . Then, the system  $\{E_k\}_{k \in \mathfrak{D}}$  forms an orthonormal basis for  $\ell^2(\mathfrak{D}^*)$ .*

*Proof.* Observe that for  $k_1, k_2 \in \mathfrak{D}$ , we have

$$\langle E_{k_1}, E_{k_2} \rangle = \sum_{m \in \mathfrak{D}^*} E_{k_1}(m) \overline{E_{k_2}(m)} = \frac{1}{|\mathfrak{D}^*|} \sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m, (Ak_1 - Ak_2) \rangle / N}. \quad (2.3)$$

We claim that

$$\sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m, (Ak_1 - Ak_2) \rangle / N} = \begin{cases} |\mathfrak{D}^*| & \text{for } k_1 = k_2, \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

For this, let  $m_1 \in \mathfrak{D}^*$  be an arbitrary element. Then, we can write

$$e^{2\pi i \langle m_1, (Ak_1 - Ak_2) \rangle / N} \sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m, (Ak_1 - Ak_2) \rangle / N} = \sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle (m+m_1), (Ak_1 - Ak_2) \rangle / N}.$$



Now, in view of Theorem 2.4, we have  $Ak_1 - Ak_2 = Ak$ , for some  $k \in \mathfrak{D}$ . Therefore, by substituting  $m + m_1 = m_2$ , we get

$$e^{2\pi i \langle m_1, Ak \rangle / N} \sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m, Ak \rangle / N} = \sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m + m_1, Ak \rangle / N} = \sum_{m_2 \in \mathfrak{D}^*} e^{2\pi i \langle m_2, Ak \rangle / N},$$

which implies that either  $e^{2\pi i \langle m_1, Ak \rangle / N} = 1$ , or,  $\sum_{m \in \mathfrak{D}^*} e^{2\pi i \langle m, Ak \rangle / N} = 0$ . But,  $e^{2\pi i \langle m_1, Ak \rangle / N} = 1$  if and only if  $\langle m_1, Ak \rangle \in N\mathbb{Z}$ , which is if and only if we have  $Ak \in N\mathbb{Z}^2$  as  $m_1 \in \mathfrak{D}^*$  is arbitrary. Equivalently, we can say that  $e^{2\pi i \langle m_1, Ak \rangle / N} = 1$  if and only if  $k \in NA^{-1}\mathbb{Z}^2 \cap \mathfrak{D} = \{(0, 0)^t\}$ , which in view of Theorem 2.4 implies that  $k_1 = k_2$ . Now, by using (2.4) and (2.3), we get  $\langle E_{k_1}, E_{k_2} \rangle = \delta_{k_1 k_2}$ . Hence the result follows.  $\square$

*Proof of Theorem 2.11.* The system  $\mathfrak{B}(\Phi)$  will form an orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  if and only if for  $k \in \mathfrak{D}$  and  $\varphi_{p_1}, \varphi_{p_2} \in \Phi$ , where  $0 \leq p_1, p_2 \leq q - 1$ , we have

$$\delta_{p_1 p_2} \delta(k) = \langle \varphi_{p_1}, T_{Ak} \varphi_{p_2} \rangle = \frac{1}{N^2} \langle \widehat{\varphi_{p_1}}, \widehat{T_{Ak} \varphi_{p_2}} \rangle = \frac{1}{N^2} \sum_{n \in \mathbb{Z}_N^2} \widehat{\varphi_{p_1}}(n) \overline{\widehat{\varphi_{p_2}}(n)} e^{2\pi i \langle n, Ak \rangle / N},$$

in view of the Plancherel's formula. Next, by applying Lemma 2.12 in the above equation, we get

$$\begin{aligned} \delta_{p_1 p_2} \delta(k) &= \frac{1}{N^2} \sum_{m \in \mathfrak{D}^*} \sum_{\gamma \in C\mathbb{Z}_N^2} \widehat{\varphi_{p_1}}(m + \gamma) \overline{\widehat{\varphi_{p_2}}(m + \gamma)} e^{2\pi i \langle (m + \gamma), Ak \rangle / N} \\ &= \frac{1}{N^2} \sum_{m \in \mathfrak{D}^*} \sum_{\gamma \in C\mathbb{Z}_N^2} \widehat{\varphi_{p_1}}(m + \gamma) \overline{\widehat{\varphi_{p_2}}(m + \gamma)} e^{2\pi i \langle m, Ak \rangle / N} \\ &= \frac{1}{q|\mathfrak{D}^*|} \sum_{m \in \mathfrak{D}^*} \Psi(m) e^{2\pi i \langle m, Ak \rangle / N}, \end{aligned}$$

where  $\Psi(m) = \sum_{\gamma \in C\mathbb{Z}_N^2} \widehat{\varphi_{p_1}}(m + \gamma) \overline{\widehat{\varphi_{p_2}}(m + \gamma)}$ , for all  $m \in \mathfrak{D}^*$ , and hence, for all  $k \in \mathfrak{D}$ , we have

$\delta_{p_1 p_2} \delta(k) = \frac{1}{q} \mathcal{F}^{-1}(\Psi(k))$ , where  $\mathcal{F}^{-1}$  denotes the Inverse discrete Fourier transform on  $\ell^2(\mathfrak{D}^*)$ . Next, by noting  $(\Psi(m))_{m \in \mathfrak{D}^*} \in \ell^2(\mathfrak{D}^*)$ , Proposition 2.13 and Lemma 2.6, we get  $\Psi(k) = q\delta_{p_1 p_2}$ . Hence, the system  $\mathfrak{B}(\Phi)$  forms an ONWS in  $\ell^2(\mathbb{Z}_N^2)$  if and only if for  $k \in \mathfrak{D}$  and  $\varphi_{p_1}, \varphi_{p_2} \in \Phi$ , where  $0 \leq p_1, p_2 \leq q - 1$ , we have

$$\mathfrak{R}_{p_1 p_2} := \sum_{\gamma \in C\mathbb{Z}_N^2} \widehat{\varphi_{p_1}}(k + \gamma) \overline{\widehat{\varphi_{p_2}}(k + \gamma)} = q\delta_{p_1 p_2} \text{ for all } k \in \mathfrak{D}.$$

Further, we note that  $\mathfrak{R}_{p_2 p_1} = \overline{\mathfrak{R}_{p_1 p_2}}$ , which proves (i)  $\Leftrightarrow$  (ii) part. Observe that above equation is equivalent to the fact that for  $k \in \mathfrak{D}$ , columns of  $\mathcal{S}_\Phi(k)$ , the system matrix of  $\Phi$  having order  $q \times q$ , defined by

$$\mathcal{S}_\Phi(k) = \frac{1}{\sqrt{q}} \left( \widehat{\varphi_p}(k + \gamma) \right)_{\substack{\gamma \in C\mathbb{Z}_N^2 \\ 0 \leq p \leq q-1}},$$

forms an orthonormal basis for  $\mathbb{C}^q$ . Equivalently, the collection  $\mathfrak{B}(\Phi) \subset \ell^2(\mathbb{Z}_N^2)$  is an ONWS in  $\ell^2(\mathbb{Z}_N^2)$  if and only if the system matrix of  $\Phi$  is unitary for each  $k \in \mathfrak{D}$ . Hence (i)  $\Leftrightarrow$  (iii) follows.  $\square$

Following are some examples of ONWS in  $\ell^2(\mathbb{Z}_N^2)$  with respect to expansive as well as non-expansive matrices. In the case of  $L^2(\mathbb{R}^n)$ , the expansive and non-expansive nature of a dilation matrix plays an important role in the existence of an orthonormal wavelet. A full characterization of dilation matrices which yields wavelets is still an open problem.

**Example 2.14.** (i) **For non-expansive matrix:** Let us recall Example 2.5 in which  $\mathfrak{D} = \{(0, 0)^t, (1, 0)^t\}$ , and the matrix  $A$  is non-expansive since one of its eigenvalues  $(2 \pm \sqrt{2})$  is less than 1. Consider  $\Phi_1 = \{\varphi_0, \varphi_1\} \subset \ell^2(\mathbb{Z}_2^2)$  defined by

$$\varphi_0 = \begin{pmatrix} \varphi_0((0, 0)^t) & \varphi_0((0, 1)^t) \\ \varphi_0((1, 0)^t) & \varphi_0((1, 1)^t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \text{ and } \varphi_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Then, the collection  $\mathfrak{B}(\Phi_1) = \{T_{Ak}\varphi_p : k \in \mathfrak{D}, 0 \leq p \leq 1\} = \{\varphi_0, \varphi_1, T_{(0,1)^t}\varphi_0, T_{(0,1)^t}\varphi_1\}$  is an ONWS for  $\ell^2(\mathbb{Z}_2^2)$ , where  $T_{(0,1)^t}\varphi_0 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$  and  $T_{(0,1)^t}\varphi_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$ . Further, the discrete Fourier transforms of  $\varphi_0$  and  $\varphi_1$  are given by  $\widehat{\varphi}_0 = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 0 \end{pmatrix}$  and  $\widehat{\varphi}_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$ , respectively. We can easily verify that for each  $k \in \mathfrak{D}$  and  $0 \leq p_1, p_2 \leq 1$ ,

$$\sum_{\gamma \in C\mathbb{Z}_2^2} \widehat{\varphi}_{p_1}(k + \gamma) \overline{\widehat{\varphi}_{p_2}(k + \gamma)} = 2\delta_{p_1 p_2},$$

where  $C\mathbb{Z}_2^2 = B^t\mathbb{Z}_2^2 = \{(0, 0)^t, (1, 0)^t\}$ . Hence,  $\mathfrak{B}(\Phi_1) = \{T_{Ak}\varphi_p : k \in \mathfrak{D}, 0 \leq p \leq 1\} \subset \ell^2(\mathbb{Z}_2^2)$  forms an ONWS in  $\ell^2(\mathbb{Z}_2^2)$ , by using Theorem 2.11.

(ii) **For expansive matrix:** Let  $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$  be an expansive matrix with eigenvalues 2, 2.

Then,  $A\mathbb{Z}_4^2 = B\mathbb{Z}_4^2 = \{(0, 0)^t, (1, 3)^t, (2, 2)^t, (3, 1)^t\}$ , where  $B = NA^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ . Further, we can choose the set of digits  $\mathfrak{D} = \{(0, 0)^t, (0, 1)^t, (0, 2)^t, (0, 3)^t\} \subset \mathbb{Z}_4^2$ , which satisfies all properties from Theorem 2.4. Next, we consider  $\Phi_2 = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3\} \subset \ell^2(\mathbb{Z}_4^2)$  whose discrete Fourier transforms are given by

$$\widehat{\varphi}_0 = \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2}i & 0 \\ -\sqrt{2}i & 0 & -\sqrt{2} & 0 \\ 0 & 1-i & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 & -1-i \end{pmatrix}, \widehat{\varphi}_1 = \begin{pmatrix} 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2}i & 0 & -\sqrt{2}i \\ -\sqrt{2} & 0 & \sqrt{2}i & 0 \\ \sqrt{2}i & 0 & \sqrt{2} & 0 \end{pmatrix},$$

$$\widehat{\varphi}_2 = \begin{pmatrix} 0 & \sqrt{2}i & 0 & \sqrt{2}i \\ 0 & -1+i & 0 & 1-i \\ -\sqrt{2}i & 0 & 1-i & 0 \\ \sqrt{2} & 0 & -\sqrt{2}i & 0 \end{pmatrix}, \text{ and } \widehat{\varphi}_3 = \begin{pmatrix} -1-i & 0 & 1-i & 0 \\ 1+i & 0 & 1-i & 0 \\ 0 & -\sqrt{2}i & 0 & -1+i \\ 0 & 1+i & 0 & -\sqrt{2}i \end{pmatrix}.$$

Then, for  $0 \leq p_1, p_2 \leq 3$ , we can check that  $\sum_{\gamma \in C\mathbb{Z}_4^2} \widehat{\varphi}_{p_1}(k + \gamma) \overline{\widehat{\varphi}_{p_2}(k + \gamma)} = 4\delta_{p_1 p_2}$  for all  $k \in \mathfrak{D}$ ,

where  $C\mathbb{Z}_4^2 = B^t\mathbb{Z}_4^2 = \{(0, 0)^t, (1, 1)^t, (2, 2)^t, (3, 3)^t\}$ . Hence, from Theorem 2.11, we conclude that the collection  $\mathfrak{B}(\Phi_2) = \{T_{Ak}\varphi_p : k \in \mathfrak{D}, 0 \leq p \leq 3\} \subset \ell^2(\mathbb{Z}_4^2)$  forms an ONWS in  $\ell^2(\mathbb{Z}_4^2)$ .

In the next section, we discuss some results on uncertainty principle by coupling Fourier basis and standard orthonormal basis with the orthonormal wavelet system  $\mathfrak{B}(\Phi)$  defined in (2.2). For this, let

us denote the standard orthonormal basis for  $\ell^2(\mathbb{Z}_N^2)$  by  $\mathfrak{B}_S := \{T_{Ak}e_{\alpha_j} : k \in \mathfrak{D}, 0 \leq j \leq q-1\}$ , where  $\{\alpha_j\}_{j=0}^{q-1} = \mathfrak{D}_0 \subset \mathbb{Z}_N^2$  (see Proposition 2.9 for more details). By  $\mathfrak{F} := \{F_v\}_{v \in \mathbb{Z}_N^2}$ , we denote a two dimensional Fourier basis for  $\ell^2(\mathbb{Z}_N^2)$  defined by  $F_v(n) = \frac{1}{N}e^{2\pi i \langle n, v \rangle / N}$ , for all  $n \in \mathbb{Z}_N^2$ . For a non-zero vector  $f \in \ell^2(\mathbb{Z}_N^2)$ , we can write

$$f = \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} t_{j,k} T_{Ak} e_{\alpha_j} = \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} s_{p,m} T_{Am} \varphi_p = \sum_{v \in \mathbb{Z}_N^2} w_v F_v, \quad (2.5)$$

and hence, we have

$$\|f\|^2 = \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |t_{j,k}|^2 = \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}|^2 = \sum_{v \in \mathbb{Z}_N^2} |w_v|^2. \quad (2.6)$$

Further, we denote the number of non-zero coefficients from among  $\{t_{j,k} : k \in \mathfrak{D}, 0 \leq j \leq q-1\}$ ,  $\{w_v : v \in \mathbb{Z}_N^2\}$  and  $\{s_{p,m} : m \in \mathfrak{D}, 0 \leq p \leq q-1\}$  by  $S_f, C_f$  and  $W_f$ , respectively. Note that  $\max X$  represents the maximum of all elements of the set  $X \subset \mathbb{R}$ .

### 3. UNCERTAINTY PRINCIPLE CORRESPONDING TO ONWS

Uncertainty principles put restrictions on how well frequency localized a good time localized signal can be and vice versa. In the case of a signal defined on a finite abelian group, localization is generally expressed through the cardinality of the support of the signal. Uniqueness of sparse representation of a signal depends upon the bound provided by uncertainty relations in terms of pair of bases. For the setup of  $\ell^2(\mathbb{Z}_N^2)$ , we prove following results on the uncertainty principle with respect to  $\mathfrak{B}(\Phi)$  that can be generalized for the case of  $\ell^2(\mathbb{Z}_N^M)$ ,  $M \in \mathbb{N}$ :

**Theorem 3.1.** *Let  $\Phi = \{\varphi_p\}_{p=0}^{q-1} \subset \ell^2(\mathbb{Z}_N^2)$  be such that  $\mathfrak{B}(\Phi)$  is an ONWS in  $\ell^2(\mathbb{Z}_N^2)$ . Consider two positive real numbers  $R_0$  and  $E_0$  defined by*

$$R_0 = \max \{|\varphi_p(\alpha + A\beta)| : \alpha \in \mathfrak{D}_0, \beta \in \mathfrak{D}, 0 \leq p \leq q-1\}, \text{ and}$$

$$E_0 = \max \left\{ \frac{1}{N} \sum_{n \in \mathbb{Z}_N^2} |\varphi_p(n)| : 0 \leq p \leq q-1 \right\}.$$

*Then, the following inequalities hold true:*

- (i) *The bounds for  $R_0$  and  $E_0$  are given by  $\frac{1}{N} \leq R_0, E_0 \leq 1$ . In case of  $R_0 = 1$ , the system  $\mathfrak{B}(\Phi)$  is not frequency localized.*
- (ii) *Representations of  $f \in \ell^2(\mathbb{Z}_N^2)$  in terms of  $\mathfrak{B}(\Phi)$  and  $\mathfrak{B}_S$  provide following relations:*

$$S_f W_f \geq \max \left\{ 2, \frac{1}{R_0^2} \right\}, \text{ and } S_f + W_f \geq \max \left\{ 3, \frac{2}{R_0} \right\}.$$

- (iii) *Representations of  $f \in \ell^2(\mathbb{Z}_N^2)$  in terms of  $\mathfrak{B}(\Phi)$  and  $\mathfrak{F}$  provide following relations:*

$$C_f W_f \geq \frac{1}{E_0^2}, \text{ and } C_f + W_f \geq \frac{2}{E_0}.$$

*Proof.* By considering representations of  $f \in \ell^2(\mathbb{Z}_N^2)$  in terms of  $\mathfrak{B}(\Phi)$  and  $\mathfrak{B}_S$  from (2.5), we have the following:

$$\begin{aligned} \|f\|^2 &= \left| \left\langle \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} s_{p,m} T_{Am} \varphi_p, \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} t_{j,k} T_{Ak} e_{\alpha_j} \right\rangle \right| = \left| \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} s_{p,m} \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} \overline{t_{j,k}} \langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle \right| \\ &\leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |s_{p,m}| |\overline{t_{j,k}}| |\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle|, \end{aligned}$$

and hence we have

$$\|f\|^2 \leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}| \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| |\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle|. \quad (3.1)$$

Similarly, if we proceed by using representations of  $f \in \ell^2(\mathbb{Z}_N^2)$  in terms of  $\mathfrak{B}(\Phi)$  and  $\mathfrak{F}$ , we get

$$\|f\|^2 \leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}| \sum_{v \in \mathbb{Z}_N^2} |\overline{w_v}| |\langle T_{Am} \varphi_p, F_v \rangle|. \quad (3.2)$$

In view of Theorem 2.4, we observe that for  $k, m \in \mathfrak{D}$  and  $0 \leq j, p \leq q-1$ ,

$$\begin{aligned} |\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle| &= |\langle \varphi_p, T_{(Ak-Am)} e_{\alpha_j} \rangle| = |\langle \varphi_p, T_{A\beta} e_{\alpha_j} \rangle| \text{ for some } \beta \in \mathfrak{D} \\ &= |\text{trace}((T_{A\beta} e_{\alpha_j})^* \varphi_p)| = \left| \sum_{n \in \mathbb{Z}_N^2} \varphi_p(n) \overline{T_{A\beta} e_{\alpha_j}(n)} \right| \\ &= \left| \sum_{n \in \mathbb{Z}_N^2} \varphi_p(n) \overline{e_{\alpha_j + A\beta}(n)} \right| = |\varphi_p(\alpha_j + A\beta)|, \end{aligned}$$

which implies that  $R_1 = R_2$ , for  $R_1 = \{|\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle| : k, m \in \mathfrak{D}, 0 \leq j, p \leq q-1\}$ , and  $R_2 = \{|\varphi_p(\alpha_j + A\beta)| : \alpha_j \in \mathfrak{D}_0, \beta \in \mathfrak{D}, 0 \leq p, j \leq q-1\}$ , and hence for any  $h \in R_1$ , we have  $h \leq R_0$ , where  $R_0 = \max R_2$ . Using this fact in (3.1), along with (2.6) and the Cauchy-Schwarz inequality, it is clear that

$$\begin{aligned} \|f\|^2 &\leq \sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}| \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| R_0 \leq \sqrt{\sum_{m,p: |s_{p,m}| \neq 0} |s_{p,m}|^2} \sqrt{\sum_{m,p: |s_{p,m}| \neq 0} \left( \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| R_0 \right)^2} \\ &= \sqrt{\sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |s_{p,m}|^2} \sqrt{W_f \left( \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| R_0 \right)^2} = \|f\| \sqrt{W_f} \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\overline{t_{j,k}}| R_0 \\ &\leq \|f\| \sqrt{W_f} \sqrt{\sum_{j,k: |t_{j,k}| \neq 0} |\overline{t_{j,k}}|^2} \sqrt{\sum_{j,k: |t_{j,k}| \neq 0} (R_0)^2} = \|f\| \sqrt{W_f} \sqrt{\sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |t_{j,k}|^2} \sqrt{S_f (R_0)^2} \\ &= \|f\|^2 \sqrt{W_f} \sqrt{S_f} R_0. \end{aligned}$$

Therefore, we have the inequality  $\sqrt{S_f W_f} \geq \frac{1}{R_0}$ . Further, by assuming part (i) [which we will prove later], we cannot consider  $R_0$  equal to 1, otherwise the system  $\mathfrak{B}(\Phi)$  will not remain time-frequency localized. Next, by using inequality of arithmetic and geometric means, we have  $S_f + W_f \geq 2\sqrt{S_f W_f} \geq \frac{2}{R_0}$ , which leads to (ii).

Next, we prove (iii) part. For this, first we observe the following:

$$|\langle T_{Am}\varphi_p, F_v \rangle| = \left| \sum_{n \in \mathbb{Z}_N^2} T_{Am}\varphi_p(n) \overline{F_v(n)} \right| = \left| \sum_{n \in \mathbb{Z}_N^2} \varphi_p(n - Am) \frac{1}{N} e^{-2\pi i \langle n, v \rangle / N} \right| \leq \frac{1}{N} \sum_{t \in \mathbb{Z}_N^2} |\varphi_p(t)|,$$

for  $m \in \mathfrak{D}, 0 \leq p \leq q-1$  and  $v \in \mathbb{Z}_N^2$ , which implies that  $\max X \leq \max Y$ , where sets  $X$  and  $Y$  are given by  $\{|\langle T_{Am}\varphi_p, F_v \rangle| : m \in \mathfrak{D}, 0 \leq p \leq q-1, v \in \mathbb{Z}_N^2\}$  and  $\left\{ \frac{1}{N} \sum_{t \in \mathbb{Z}_N^2} |\varphi_p(t)| : 0 \leq p \leq q-1 \right\}$ ,

respectively, and hence for any  $x \in X$ , we have  $x \leq E_0$ , where  $E_0 = \max Y$ . Therefore, we obtain the inequality  $\sqrt{C_f W_f} \geq \frac{1}{E_0}$ , using the above discussion in (3.2) along with the way we proceeded in case of (ii). Now, by inequality of arithmetic and geometric means, we get  $C_f + W_f \geq 2\sqrt{C_f W_f} \geq \frac{2}{E_0}$ , and hence (iii) follows.

For the part (i), let us consider elements of  $\mathfrak{B}(\Phi)$  and  $\mathfrak{B}_S$ . Then, from Cauchy-Schwarz inequality, we have

$$|\langle T_{Am}\varphi_p, T_{Ak}e_{\alpha_j} \rangle| \leq \|T_{Am}\varphi_p\| \|T_{Ak}e_{\alpha_j}\| = 1,$$

for all  $k, m \in \mathfrak{D}$  and  $0 \leq j, p \leq q-1$ . Therefore, 1 is an upper bound for the set  $R_1$  and hence its maximum element  $R_0 \leq 1$ . Next, we note that the real number  $R_0$  cannot take the value 1. For this, let by contradiction we assume  $R_0 = 1$ . Then, we have

$$\begin{aligned} 1 &= \max \{|\varphi_p(\alpha + A\beta)| : \alpha \in \mathfrak{D}_0, \beta \in \mathfrak{D}, 0 \leq p \leq q-1\} \\ &= \max \{|\varphi_p(n)| : n \in \mathbb{Z}_N^2, 0 \leq p \leq q-1\}, \end{aligned}$$

since the collection  $\{\alpha + Ak : \alpha \in \mathfrak{D}_0, k \in \mathfrak{D}\}$  is a partition of  $\mathbb{Z}_N^2$ . This assures the existence of  $p_1 \in \{0, 1, \dots, q-1\}$  and  $n_1 \in \mathbb{Z}_N^2$  such that  $|\varphi_{p_1}(n_1)| = 1$ . Therefore, we have

$$\|\varphi_{p_1}\|^2 = \sum_{m \in \mathbb{Z}_N^2} |\varphi_{p_1}(m)|^2 = |\varphi_{p_1}(n_1)|^2 + \sum_{m \neq n_1 \in \mathbb{Z}_N^2} |\varphi_{p_1}(m)|^2 = 1, \quad (3.3)$$

in view of the fact that  $\varphi_{p_1} \in \Phi$  and  $\mathfrak{B}(\Phi)$  forms an ONWS in  $\ell^2(\mathbb{Z}_N^2)$ . Since  $|\varphi_{p_1}(n_1)| = 1$ , therefore from (3.3), it is clear that  $\sum_{m \neq n_1 \in \mathbb{Z}_N^2} |\varphi_{p_1}(m)|^2 = 0$ , which provides  $\varphi_{p_1}(m) = 0$  for all  $m \neq n_1 \in \mathbb{Z}_N^2$ ,

and hence we can define  $\varphi_{p_1} \in \Phi \subset \ell^2(\mathbb{Z}_N^2)$  by  $\varphi_{p_1}(n) = e^{i\theta(n)}$  for  $n = n_1 \in \mathbb{Z}_N^2$ , and zero otherwise, where the real number  $\theta(n)$  depends on  $n$ . Therefore, we have  $\widehat{\varphi_{p_1}}(m) = \sum_{n \in \mathbb{Z}_N^2} \varphi_{p_1}(n) e^{-2\pi i \langle m, n \rangle / N} =$

$e^{i\theta(n_1)} e^{-2\pi i \langle m, n_1 \rangle / N}$ , and hence  $|\widehat{\varphi_{p_1}}(m)| = 1$  for all  $m \in \mathbb{Z}_N^2$ . Thus, we conclude that  $\varphi_{p_1}$  is not frequency localized which implies that the system  $\mathfrak{B}(\Phi)$  is not time-frequency localized.

Now, for computing a lower bound of  $R_0$  (greater than zero), we consider an  $N^2 \times N^2$  matrix  $M = (\langle T_{Am}\varphi_p, T_{Ak}e_{\alpha_j} \rangle)$ , where rows of the matrix are varying over  $m \in \mathfrak{D}$  and  $0 \leq p \leq q-1$ , and

columns over  $k \in \mathfrak{D}$  and  $0 \leq j \leq q-1$ . Observe that, for each  $k, m \in \mathfrak{D}$  and  $0 \leq p, j \leq q-1$ , we have

$$\sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} |\langle T_{Am} \varphi_p, T_{Ak} e_{\alpha_j} \rangle|^2 = \|T_{Ak} e_{\alpha_j}\|^2 = 1, \text{ and } \sum_{k \in \mathfrak{D}} \sum_{j=0}^{q-1} |\langle T_{Ak} e_{\alpha_j}, T_{Am} \varphi_p \rangle|^2 = \|T_{Am} \varphi_p\|^2 = 1.$$

Therefore, rows and columns of  $M$  have unit norm. Further, for  $0 \leq j_1 \neq j_2, p \leq q-1$  and  $m, k_1 \neq k_2 \in \mathfrak{D}$ , the inner product of any two columns of  $M$  given by

$$\sum_{m \in \mathfrak{D}} \sum_{p=0}^{q-1} \langle T_{Am} \varphi_p, T_{Ak_1} e_{j_1} \rangle \overline{\langle T_{Am} \varphi_p, T_{Ak_2} e_{j_2} \rangle} = \langle T_{Ak_2} e_{j_2}, T_{Ak_1} e_{j_1} \rangle = 0,$$

implies that columns of  $M$  are orthogonal. Similarly, we can check rows of  $M$  are orthogonal, and hence  $M$  is an orthonormal matrix having the sum of squares of its entries equal to  $N^2$ . Therefore, all of its entries cannot be less than  $1/N$ . This follows by noting that if we assume that all entries of  $M$  are less than  $1/N$ , then, sum of squares of all  $N^4$  entries of  $M$  will be less than  $N^4 \times \frac{1}{N^2}$ , that is,  $N^2$ , which is not true. Further, observe that the absolute values of all entries of matrix  $M$  are exactly the elements of the set  $R_1$ . Hence, we conclude that there exists  $h \in R_1$  such that  $h \geq 1/N$ . Thus,  $R_0 \geq 1/N$ . Hence, we have  $1/N \leq R_0 < 1$ .

Similarly, we can show that  $1/N \leq E_0 \leq 1$  by considering elements of  $\mathfrak{B}(\Phi)$  and  $\mathfrak{F}$ .  $\square$

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